



Egyptian Mathematical Society
Journal of the Egyptian Mathematical Society

www.etms-eg.org
 www.elsevier.com/locate/joems



SHORT COMMUNICATION

Existence of uniformly stable solutions of nonautonomous discontinuous dynamical systems

A.M.A. El-Sayed ^{a,*}, M.E. Nasr ^b

^a Faculty of Science, Alexandria University, Alexandria, Egypt

^b Faculty of Science, Benha University, Benha 13518, Egypt

Available online 9 November 2011

Abstract We are concerned here with the existence of uniformly Lyapunov stable integrable solution of linear and nonlinear nonautonomous discontinuous dynamical systems.

© 2011 Egyptian Mathematical Society. Production and hosting by Elsevier B.V. All rights reserved.

1. Introduction

The discrete dynamical system [1,4]

$$x_n = ax_{n-1}, \quad n = 1, 2, \dots \quad (1)$$

$$x_0 = c. \quad (2)$$

has the discrete solution

$$x_n = a^n x_0, \quad n = 1, 2, \dots \quad (3)$$

The more general dynamical system

$$x(t) = ax(t-r), \quad t \in (0, T] \text{ and } r > 0 \quad (4)$$

$$x(t) = x_0, \quad t \leq 0. \quad (5)$$

has the discontinuous (integrable) solution

$$x(t) = a^{1+\lfloor t/r \rfloor} x_0 \in L_1(0, T], \quad \text{where } [\cdot] \text{ is the fract function.} \quad (6)$$

The nonlinear discrete dynamical system

$$x_n = f(x_{n-1}), \quad n = 1, 2, \dots \quad (7)$$

with the initial data (2)

$$x_0 = c$$

has the discrete solution

$$x_n = f^n(x_0), \quad n = 1, 2, \dots \quad (8)$$

but the nonlinear problem

$$x(t) = f(x(t-r)), \quad r > 0 \quad (9)$$

with the initial data (5) is more general than the problem (7)–(14) and has the discontinuous (integrable) solution

$$x_n = f^{1+\lfloor t/r \rfloor}(x_0) \in L_1(0, T) \quad (10)$$

So, we can call the systems (4)–(5) and (9)–(10) are discontinuous dynamical systems.

Definition 1. The discontinuous dynamical system is the problem of the retarded functional equation

* Corresponding author.

E-mail addresses: amasayed5@yahoo.com, amasayed@hotmail.com (A.M.A. El-Sayed).

1110-256X © 2011 Egyptian Mathematical Society. Production and hosting by Elsevier B.V. All rights reserved.

Peer review under responsibility of Egyptian Mathematical Society.
 doi:10.1016/j.joems.2011.09.006



Production and hosting by Elsevier

$$x(t) = f(t, x(t-r)), \quad r, \quad t > 0 \quad (11)$$

$$x(t) = g(t), \quad t \in (-\infty, 0]. \quad (12)$$

Let $L_1[a, b]$, $-\infty < a < b < \infty$ be the class of Lebesgue integrable function defined on $[a, b]$ with the equivalent norm

$$\|x\|_{L_1[a,b]} = \int_a^b e^{-Nt} |x(t)| dt, \quad x \in L_1[a, b], \quad N > 0 \text{ is arbitrary}$$

and let ${}_nL_1[a, b]$, $-\infty < a < b < \infty$ be the class of Lebesgue integrable column vectors $X(t) = (x_1(t), x_2(t), \dots, x_n(t))'$ defined on $[a, b]$ with the equivalent norm

$$\|X\|_{{}_nL_1[a,b]} = \sum_{k=0}^n \int_a^b e^{-Nt} |x_k(t)| dt, \quad x_k \in L_1[a, b], \quad N > 0 \text{ is arbitrary}$$

Here we prove the existence of a unique uniformly Lyapunov stable solution $x \in L_1[0, T]$ for the nonautonomous nonlinear discontinuous dynamical system.

$$x_i(t) = f_i(t, x_1(t-r_1), x_2(t-r_2), \dots, x_n(t-r_n)), \quad t, \quad r_i > 0 \quad (13)$$

$$x_i(t) = g_i(t) \in L_1(-r, 0], \quad \lim_{t \rightarrow 0} g_i(t) = x_i(0), \quad i = 1, 2, \dots, n \quad (14)$$

where $r = \min\{r_1, r_2, \dots, r_n\}$.

The nonautonomous linear discontinuous dynamical system

$$x_i(t) = \sum_{j=1}^n a_{ij} x_j(t) + \sum_{j=1}^n b_{ij}(t) x_j(t-r_j) \quad (15)$$

$$x_i(t) = g_i(t) \in L_1(-r, 0], \quad \lim_{t \rightarrow 0} g_i(t) = x_i(0)$$

where a_{ij} and $r_j > 0$, $r = \min\{r_1, r_2, \dots, r_n\}$ are constants and b_{ij} are bounded functions on $(0, T]$, $T < \infty$ for $i, j = 1, 2, \dots, n$ will be studied.

2. Nonlinear systems

Consider the nonlinear nonautonomous nonlinear discontinuous dynamical system (13) and (14).

Theorem 1. *If the functions f_i , $i = 1, 2, \dots, n$ are continuous on $[0, T]$ and satisfy the Lipschitz condition*

$$|f_i(t, x_1, x_2, \dots, x_n) - f_i(t, y_1, y_2, \dots, y_n)| \leq k_i \sum_{j=1}^n |x_j - y_j|,$$

then the discontinuous dynamical system (13) and (14) has a unique integrable solution $x \in L_1(0, T]$, $T < \infty$.

This solution is uniformly Lyapunov stable in the sense that $\forall \epsilon > 0$, there exists $\delta > 0$ such that

$$\|G - G^*\| = \sum_{i=1}^n \|g_i - g_i^*\|_{L_1(-r, 0]} < \delta, \quad \text{implies } \|X - X^*\|_{{}_nL_1(0, T]} < \epsilon.$$

where $G = (g_i)_{n \times 1}$, $G^* = (g_i^*)_{n \times 1}$, $X = (x_i)_{n \times 1}$ and $X^* = (x_i^*)_{n \times 1}$.

Proof. Define the map $F = (F_i)_{n \times 1}$ where $F_i: L_1[0, T] \rightarrow L_1[0, T]$, $i = 1, 2, \dots, n$ are defined by

$$F_i x_i(t) = f_i(t, x_1(t-r_1), x_2(t-r_2), \dots, x_n(t-r_n)),$$

we find that

$$|F_i x_i(t) - F_i y_i(t)| \leq k_i \sum_{j=1}^n |x_j(t-r_j) - y_j(t-r_j)|$$

and

$$\begin{aligned} \|F_i x_i - F_i y_i\|_{L_1(0, T]} &\leq k_i \sum_{j=1}^n e^{-Nr_j} \int_0^T e^{-N(t-r_j)} |x_j(t-r_j) - y_j(t-r_j)| dt \\ &= k_i \sum_{j=1}^n e^{-Nr_j} \int_{r_j}^T e^{-N(t-r_j)} |x_j(t-r_j) - y_j(t-r_j)| dt \\ &= k_i \sum_{j=1}^n e^{-Nr_j} \int_0^{T-r_j} e^{-N(s_j)} |x_j(s_j) - y_j(s_j)| ds \\ &\leq k_i \sum_{j=1}^n e^{-Nr_j} \int_0^T e^{-N(\tau)} |x_j(\tau) - y_j(\tau)| d\tau \\ &\leq \left(k_i \sum_{j=1}^n e^{-Nr_j} \right) \|X - Y\|_{{}_nL_1(0, T]}. \end{aligned}$$

This implies that

$$\begin{aligned} \|FX - FY\|_{{}_nL_1(0, T]} &= \sum_{i=1}^n \|F_i x_i - F_i y_i\|_{L_1(0, T]} \\ &\leq \left(\sum_{i=1}^n k_i \sum_{j=1}^n e^{-Nr_j} \right) \|X - Y\|_{{}_nL_1(0, T]}. \end{aligned}$$

Choose N so large enough such that $(\sum_{i=1}^n k_i \sum_{j=1}^n e^{-Nr_j}) < 1$ we deduce by the contraction fixed point theorem that there exists a unique solution $x \in L_1(0, T]$ of the problem (13)–(14) [3].

Consider now the problem consisting of Eq. (13) and the initial data

$$\begin{aligned} x_i(t) &= g_i^*(t) \in L_1(-r, 0], \quad \lim_{t \rightarrow 0} g_i^*(t) = x_i^*(0), \quad i \\ &= 1, 2, \dots, n. \end{aligned} \quad (16)$$

Then we have

$$\begin{aligned} \|x_i - x_i^*\| &\leq k_i \sum_{j=1}^n e^{-Nr_j} \int_0^T e^{-N(t-r_j)} |x_j(t-r_j) - y_j(t-r_j)| dt \\ &= k_i \sum_{j=1}^n e^{-Nr_j} \left[\int_0^{r_j} e^{-N(t-r_j)} |g_j(t-r_j) - g_j^*(t-r_j)| dt \right. \\ &\quad \left. + \int_0^T e^{-N(t-r_j)} |x_j(t-r_j) - y_j(t-r_j)| dt \right] \\ &= k_i \sum_{j=1}^n e^{-Nr_j} \left[\int_{-r_j}^0 e^{-N\tau_j} |g_j(\tau_j) - g_j^*(\tau_j)| d\tau \right. \\ &\quad \left. + \int_0^{T-r_j} e^{-N(s_j)} |x_j(s_j) - x_j^*(s_j)| ds \right] \\ &\leq k_i \sum_{j=1}^n e^{-Nr_j} \left[\int_{-r}^0 e^{-N\theta} |g_j(\theta) - g_j^*(\theta)| d\theta \right. \\ &\quad \left. + \int_0^T e^{-N(u)} |x_j(u) - x_j^*(u)| du \right] \\ &= k_i \sum_{j=1}^n e^{-Nr_j} \left[\|g_j - g_j^*\|_{L_1(-r, 0]} + \|x_j - x_j^*\|_{L_1(0, T]} \right] \\ &\leq \left(k_i \sum_{j=1}^n e^{-Nr_j} \right) [\|G - G^*\|_{L_1(-r, 0]} + \|X - X^*\|_{{}_nL_1(0, T]}]. \end{aligned}$$

This implies that

$$\|X - X^*\|_{nL_1(0,T]} \leq \left(\sum_{i=1}^n k_i \sum_{j=1}^n e^{-Nr_j} \right) \left[\|G - G^*\|_{L_1(-r,0]} + \|X - X^*\|_{nL_1(0,T]} \right]. \quad (17)$$

Hence

$$\|X - X^*\|_{nL_1(0,T]} \leq \left(1 - \sum_{i=1}^n k_i \sum_{j=1}^n e^{-Nr_j} \right)^{-1} \left(\sum_{i=1}^n k_i \sum_{j=1}^n e^{-Nr_j} \right) \|G - G^*\|_{L_1(-r,0]} \quad (18)$$

and the result follows. Consider the problem consisting of Eq. (13)

$$x_i(t) = f_i(t, x_1(t-r_1), x_2(t-r_2), \dots, x_n(t-r_n)), \quad t, r_i > 0$$

and the initial data

$$x_i(t) = x_{i0}, \quad t \leq 0, \quad i = 1, 2, \dots, n. \quad (19)$$

Then the following corollary can be proved. \square

Corollary 1. *If the assumptions of Theorem 1 are satisfied, then the discontinuous dynamical system (13) and (19) has a unique integrable solution $x \in L_1(0, T]$, $T < \infty$.*

This solution is uniformly Lyapunov stable [2] in the sense that $\forall \epsilon > 0$, there exists $\delta > 0$ such that

$$\|X_o - X_o^*\|^* = \sum_{i=1}^n |x_{io} - x_{io}^*| < \delta, \text{ implies } \|X - X^*\|_{nL_1(0,T]} < \epsilon.$$

where $X_o = (x_{i o})_{n \times 1}$ and $X_o^* = (x_{i o}^*)_{n \times 1}$.

Corollary 2. *If the function f satisfies the Lipschitz condition*

$$|f(t, x_1, x_2, \dots, x_n) - f_i(t, y_1, y_2, \dots, y_n)| \leq k \sum_{j=1}^n |x_j - y_j|,$$

then the discontinuous dynamical system

$$x(t) = f(t, x(t-r_1), x(t-r_2), \dots, x(t-r_n)), \quad t, r_i > 0 \quad (20)$$

$$x(t) = g(t) \in L_1(-r, 0], \quad \lim_{t \rightarrow 0} g(t) = x(0), \quad r = \min\{r_1, r_2, \dots, r_n\} \quad (21)$$

has a unique integrable solution $x \in L_1(0, T]$. This solution is uniformly Lyapunov stable.

Example 1. Let ρ be a bounded function on $(0, T]$. The nonautonomous nonlinear discontinuous dynamical system of the Logistic equation

$$x(t) = \rho(t)x(t-r_1)(1-x(t-r_2)), \quad t > 0 \text{ and } r_2 > r_1 > 0 \quad (22)$$

with the initial data

$$x(t) = g(t) \in L_1(-r_1, 0], \quad \lim_{t \rightarrow 0} g(t) = x(0)$$

has a unique integrable solution $x \in L_1(0, T]$. This solution is uniformly Lyapunov stable.

3. Linear systems

Consider the linear nonautonomous nonlinear discontinuous dynamical system of the equation

$$x_i(t) = \sum_{j=1}^n b_{ij}(t)x_j(t-r_j) \quad (23)$$

with the data (14)

$$x_i(t) = g_i(t) \in L_1(-r, 0], \quad \lim_{t \rightarrow 0} g_i(t) = x_i(0)$$

where $r_j > 0$, $r = \min\{r_1, r_2, \dots, r_n\}$ are constants and b_{ij} are bounded functions on $(0, T]$, $T < \infty$ for $i, j = 1, 2, \dots, n$.

Let f_i in (13) be given by

$$f_i(t, x_1(t-r_1), x_2(t-r_2), \dots, x_n(t-r_n)) = \sum_{j=1}^n b_{ij}(t)x_j(t-r_j).$$

Then, by the same way as in Theorem 1, we can prove the following theorem

Theorem 2. *Let $b_{ij}(t)$ are bounded on $(0, T]$. Then the discontinuous dynamical system (23) and (14) has a unique integrable solution $x \in L_1(0, T]$, $T < \infty$. This solution is uniformly Lyapunov stable.*

Consider now the discontinuous dynamical system (15)–(14)

$$x_i(t) = \sum_{j=1}^n a_{ij}x_j(t) + \sum_{j=1}^n b_{ij}(t)x_j(t-r_j)$$

$$x_i(t) = g_i(t) \in L_1(-r, 0], \quad \lim_{t \rightarrow 0} g_i(t) = x_i(0)$$

where a_{ij} and $r_j > 0$, $r = \min\{r_1, r_2, \dots, r_n\}$ are constants and b_{ij} are bounded functions on $(0, T]$, $T < \infty$ for $i, j = 1, 2, \dots, n$ will be studied.

As a corollary of Theorem 2, we can prove the following theorem for the discontinuous dynamical system (15) and (14).

Theorem 3. *Let $b_{ij}(t)$ are bounded on $(0, T]$. If the matrix $I - A$, $A = (a_{ij})$, $i, j = 1, 2, \dots, n$ is nonsingular matrix, then the discontinuous dynamical system (15)–(14) has a unique integrable solution $x \in L_1(0, T]$, $T < \infty$. This solution is uniformly Lyapunov stable.*

Proof. Let $B(t) = (b_{ij}(t))_{n \times n}$ and $C(t) = (I - A)^{-1}B(t) = \left(\sum_{i,j=1}^n a_{ij}b_{ij}(t) \right) = (c_{ij}(t))$, then $c_{ij}(t)$ are bounded on $(0, T]$.

Let $'$ means the transpose of the matrix and write (15) in the form

$$X(t) = AX(t) + B(t)(x_1(t-r_1), x_2(t-r_2), \dots, x(t-r_n))',$$

then

$$(I - A)X(t) = B(t)(x_1(t-r_1), x_2(t-r_2), \dots, x(t-r_n))',$$

which implies that

$$X(t) = (I - A)^{-1}B(t)(x_1(t-r_1), x_2(t-r_2), \dots, x(t-r_n))'$$

and (15) can be written as

$$x_i(t) = \sum_{j=1}^n c_{ij}(t)x_j(t-r_j)$$

and the proof follows from Theorem 2. \square

Corollary 3. *The discontinuous dynamical system (15)–(19)*

$$x_i(t) = \sum_{j=1}^n a_{ij}x_j(t) + \sum_{j=1}^n b_{ij}(t)x_j(t-r_j)$$

$$x_i(t) = x_{i0}, \quad t \leq 0, \quad i = 1, 2, \dots, n$$

has a unique integrable solution $x \in L_1(0, T]$, $T < \infty$. This solution is uniformly Lyapunov stable.

References

- [1] S. Elaydi, An Introduction to Difference Equations, third ed., Springer, 2004.
- [2] G.A. Coddington, N. Levinson, Theory of Differential Equations, McGraw-Hill Book Company, New York, 1955.
- [3] R.F. Curtain, A.J. Pritchard, Functional Analysis in Modern Applied Mathematics, Academic press, 1977.
- [4] Oded. Galor, Discrete Dynamical System, Springer, 2007.